ELECTROMAGNETIC DISTURBANCE CAUSED BY EXPANDING IDEALLY CONDUCTING SPHERE IN A MAGNETIC FIELD

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We examine the electromagnetic disturbance which occurs during linear expansion of an ideally conducting sphere in an external homogeneous magnetic field for velocities comparable with the speed of light. We examine the field transition into the static regime after the sphere stops.

1. The problem of the electromagnetic fields which develop during rapid expansion of an ideally conducting sphere in a homogeneous magnetic field was studied in [1], where an erroneous result was obtained (the solution does not satisfy the basic Maxwell equations). This remark also applies to the results presented in [1] for a pulsating sphere.

Assume that an ideally conducting sphere whose center coincides with the center of the spherical coordinate system expands following the linear law $a = vt$ (a is the sphere radius) in an external uniform magnetic field $\mathbf{H}_0$ directed along the z axis.

The electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ satisfy the equations

$$\operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \quad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad \operatorname{div}\mathbf{H} = 0 \tag{1.1}$$

By virtue of problem symmetry, the field is independent of the angle φ, and the magnetic field has nonzero components $H_r(r, \vartheta, t)$ and $H_\vartheta(r, \vartheta, t)$, while the electric field has only $E_\varphi(r, \vartheta, t)$.

The system of equations (1.1) must be supplemented by boundary and initial conditions. It is clear from the problem formulation that at the initial time

$$H_r(r, \vartheta, 0) = H_0\cos\vartheta, \quad H_\vartheta(r, \vartheta, 0) = -H_0\sin\vartheta, \quad E_\varphi(r, \vartheta, 0) = 0 \tag{1.2}$$

In accordance with [2] the boundary condition at the sphere surface must have the form

$$(\mathbf{E} + c^{-1}[\mathbf{v}, \mathbf{H}])_{r=a} = 0 \tag{1.3}$$

or in terms of the components

$$E_\varphi(a, \vartheta, t) + a^{\cdot}c^{-1}H_\vartheta(a, \vartheta, t) = 0 \tag{1.4}$$

The angular dependence in (1.1), (1.2), (1.4) separates if we set

$$H_r(r, \vartheta, t) = \cos\vartheta\, H_r(r, t), \quad H_\vartheta(r, \vartheta, t) = -\sin\vartheta\, H_\vartheta(r, t), \quad E_\varphi(r, \vartheta, t) = \sin\vartheta\, E_\varphi(r, t), \tag{1.5}$$

after which we obtain the problem

$$\frac{1}{r}\frac{\partial(rH_\vartheta)}{\partial r} - \frac{H_r}{r} = -\frac{1}{c}\frac{\partial E_\varphi}{\partial t}, \quad \frac{2E_\varphi}{r} = -\frac{1}{c}\frac{\partial H_r}{\partial t}, \quad H_\vartheta = \frac{1}{2r}\frac{\partial}{\partial r}(r^2 H_r) \tag{1.6}$$

$$H_r(r, 0) = H_\vartheta(r, 0) = H_0, \quad E_\varphi(r, 0) = 0, \quad E_\varphi(a, t) - a^{\cdot}c^{-1}H_\vartheta(a, t) = 0 \tag{1.7}$$

2. Let us examine the solution of (1.6), (1.7) in the region

$$D = \{a(t) \leqslant r < \infty, \quad t \geqslant 0\}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 10, No. 6, pp. 102-104, November-December, 1969. Original article submitted October 10, 1968.

From (1.6) we have

$$\frac{\partial^2 H_r}{\partial r^2}+\frac{4}{r}\frac{\partial H_r}{\partial r}-\frac{1}{c^2}\frac{\partial^2 H_r}{\partial t^2}=0 \quad (2.1)$$

We shall seek the solution of (2.1), regular at infinity and satisfying the radiation principle, by separating the self-similar parameter – the method used in [3].

The solution of (2.1) must have the form of a diverging wave; therefore, we convert to the new variables

$$\rho=r, \qquad \tau=t-r/c \quad (2.2)$$

After which (2.1) takes the form

$$\frac{\partial^2 H_r}{\partial \rho^2}-\frac{2}{c}\frac{\partial^2 H_r}{\partial \rho\,\partial \tau}+\frac{4}{\rho}\frac{\partial H_r}{\partial \rho}-\frac{4}{c\rho}\frac{\partial H_r}{\partial \tau}=0 \quad (2.3)$$

In this problem there are three independent similarity criteria: H_r/H_0, $\beta=v/c$, $x=c\tau/\rho$. Therefore, on the basis of the π-theorem [4], the solution of (2.3) can be written in the form

$$H_r/H_0=f(\beta, x) \quad (2.4)$$

or, since β = const,

$$H_r(\rho, \tau)=H_0\Psi(x)$$

For $\Psi(x)$ we obtain from (2.3)

$$(x^2+2x)\frac{d^2\Psi}{dx^2}-2(x+1)\frac{d\Psi}{dx}=0 \quad (2.5)$$

having the solution

$$\Psi(x)=A(x^3+3x^2)+B \quad (2.6)$$

where A and B are constants of integration.

Returning in (2.6) to the original variables, we obtain

$$H_r(r, t)=H_0\left\{A\left[\left(\frac{ct-r}{r}\right)^3+3\left(\frac{ct-r}{r}\right)^2\right]+B\right\} \quad (2.7)$$

Substituting (2.7) into (1.6), we obtain

$$E_\varphi(r, t)=-1.5H_0A\left[\left(\frac{ct-r}{r}\right)^2+2\left(\frac{ct-r}{r}\right)\right] \quad (2.8)$$

$$H_\vartheta(r, t)=H_0\left\{\frac{A}{2}\left[\left(\frac{ct-r}{r}\right)^3+3\left(\frac{ct-r}{r}\right)^2+6\left(\frac{ct-r}{r}\right)\right]-B\right\} \quad (2.9)$$

The constants A and B are found from the conditions (1.7):

$$A=-2\beta^3(1-\beta)^{-2}(1+2\beta)^{-1}, \qquad B=1 \quad (2.10)$$

We write the solution:

$$H_r(r, \vartheta, t)=H_0\cos\vartheta\left\{1-2F(\beta)\eta(ct-r)\left[\left(\frac{vt-\beta r}{r}\right)^3+3\beta\left(\frac{vt-\beta r}{r}\right)^2\right]\right\}$$
$$H_\vartheta(r, \vartheta, t)=-H_0\sin\vartheta\left\{1+F(\beta)\eta(ct-r)\left[\left(\frac{vt-\beta r}{r}\right)^3+3\beta\left(\frac{vt-\beta r}{r}\right)^2+6\beta^2\left(\frac{vt-\beta r}{r}\right)\right]\right\} \quad (2.11)$$

$$E_\varphi(r, \vartheta, t)=3H_0\sin\vartheta\beta F(\beta)\left[\left(\frac{vt-\beta r}{r}\right)^2+2\beta\left(\frac{vt-\beta r}{r}\right)\right]\eta(ct-r)$$
$$F(\beta)=2^{-1}(1-\beta)^{-2}(1+2\beta)^{-1}$$

Here $\eta(ct-r)$ is the unit function.

We can see by direct substitution that the expressions (2.11) for the fields satisfy the Maxwell equations and the boundary and initial conditions.

3. The results obtained may be of interest in problems of rapid development of a conducting region in external fields, for example, in examining electrodynamic effects accompanying the expansions of the shock wave of a light spark in a laser beam in a magnetic field [5, 6].

At the initial moment the strong shock wave front has high conductivity, the external field does not penetrate inside the wave, and the boundary condition (1.4) is satisfied at the shock wave surface itself. In the course of time the front temperature decreases, and the surface at which (1.4) is satisfied lags behind the front and slows down rapidly.

In this connection, let us examine the following model problem: an ideally conducting sphere expands linearly in an external magnetic field up to the moment t_0; then for $t > t_0$ the sphere stops. In this case the field distribution given by (2.11) can be taken as initial distribution for $t = t_0$. Let us examine the relaxational process of field approach to the static value.

In place of the functions H_r, H_ϑ, E_φ, we introduce the function u(r, t), using the formulas

$$H_r(r, t) = \frac{2u}{r}, \qquad H_\vartheta = \frac{1}{r}\frac{\partial (ru)}{\partial r}, \qquad E_\varphi = \frac{1}{c}\frac{\partial u}{\partial t} \tag{3.1}$$

satisfying the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2u}{r^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{3.2}$$

the boundary condition

$$u(a_0, t) = 0 \tag{3.3}$$

and the initial condition

$$u(r, t_0) = \frac{rH_0}{2}\left\{1 - 2F(\beta)\left[\left(\frac{a_0 - \beta r}{r}\right)^3 + 3\beta\left(\frac{a_0 - \beta r}{r}\right)^2\right]\right\} \tag{3.4}$$

As $t \to \infty$, we can set in (3.2) $\partial^2 u/\partial t^2 = 0$, and in this limiting case we obtain

$$u(r, \infty) = {}^1/_2 rH_0(1 - a_0^3 / r^3) \tag{3.5}$$

We convert in (3.2) from the variables r, t to the variables ρ, τ, using the formulas

$$\rho = r / a_0, \qquad \tau = a_0^{-1}[c(t - t_0) - (r - a_0)]$$

Then the equation, boundary and initial conditions take the form

$$\frac{\partial^2 u}{\partial \rho^2} - 2\frac{\partial^2 u}{\partial \rho \partial \tau} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} - \frac{2}{\rho}\frac{\partial u}{\partial \tau} - \frac{2u}{\rho^2} = 0 \tag{3.6}$$

$$u(1, \tau) = 0, \qquad u(\rho, 0) = \frac{\rho a_0 H_0}{2}\left[1 - 2F(\beta)\left(\frac{1}{\rho^3} + \frac{3\beta}{\rho^2}\right)\right], \qquad u(\rho, \infty) = {}^1/_2 \rho a_0 H_0 (1 - \rho^{-3}) \tag{3.7}$$

This problem is solved by the Laplace transformation with respect to the variable τ. Omitting the tedious calculations, we write the solution

$$u(\rho, \tau) = \frac{\rho a_0 H_0}{2}\left\{1 - \frac{1}{\rho^3}[2F(\beta)e^{-\tau} + 1 - e^{-\tau}] + \frac{6\beta F(\beta)}{\rho^2}e^{-\tau}\right\} \tag{3.8}$$

Returning in (3.8) to r and t, we obtain

$$u(r, t) = \frac{rH_0}{2}\left\{1 - \left[2F(\beta)\exp\frac{-c(t - t_0) + (r - a_0)}{a_0} + 1 - \exp\frac{-c(t - t_0) + (r - a_0)}{a_0}\right]\frac{a_0^3}{r^3} + 6\beta F(\beta)\frac{a_0^2}{r^2}\exp\frac{-c(t - t_0) + (r - a_0)}{a_0}\right\} \tag{3.9}$$

This relation together with (3.1) yields the solution of the problem of the field relaxation process as the sphere stops. We see from (3.9) and (3.1) that the fields approach the static regime (the electric field approaches zero) exponentially, without oscillations. We can see from simple physical considerations that this is a consequence of the homogeneity of the external field, when during expansion of the sphere surface currents are induced at its surface which are distributed symmetrically relative to the diametral plane perpendicular to the magnetic field, and as a consequence of this symmetry transfer of energy from one pole to the other is not possible. In a nonuniform field the current distribution symmetry is disrupted and at the moment of stopping an effective ring current flows on one of the hemispheres, which induces on the other hemisphere an effective current, which in turn induces a current on the first hemisphere, and so on, as a result of which oscillations develop.

The authors wish to thank G. V. Fedorovich for helpful discussions.

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